NAME:____________**KEY**_____________ **ALPHA NUM:**___________
INSTRUCTOR: _____________________________ **SECTION:** ______________

CALCULUS II (SM122, SM122A) FINAL EXAMINATION

0755-1055 13 May 2010 SHOW ALL WORK IN THIS TEST PACKAGE

Tear off the last sheet of this exam. It contains problems 27, 28, 29, 30. Do these problems first WITHOUT YOUR CALCULATOR. When you are finished with these 4 problems, return the sheet to your instructor, take out your calculator, and complete the rest of the exam.

PART ONE: MULTIPLE CHOICE (50%). The first 20 problems are multiple-choice. Fill in the best answer on your Scantron bubble sheet. Write your name, alpha number, instructor, and section number on this test and your bubble sheet and bubble in your alpha number. There is no extra penalty for wrong answers on the multiple choice. Show all your scratch work on this test.

CALCULATORS ARE PERMITTED FOR THIS MULTIPLE CHOICE SECTION.

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1. A standard substitution converts the integral $\int x^2 e^{x^3+1} dx$ to which of the following?
a. $\int ue^u du$ b. $\int e^u du$ c. $\int \frac{1}{3} ue^u du$ d. $\int \frac{1}{3} e^u du$ e. $\int 3ue^u du$

Letting $u = x^3 + 1$ we have $\frac{du}{dx} = 3x^2$ or $\frac{1}{3} du = x^2 dx$ so we get d. $\int \frac{1}{3} e^u du$. [d]

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2. By the first part of the Fundamental Theorem of Calculus, the derivative of
$g(z) = \int_{\pi}^{z} \sqrt{1 + \cos(t)}\, dt$ is
a. $\sin(z)\sqrt{1 + \cos(z)}$ b. $\sqrt{1 + \cos(z)}$ c. $-\sin(z)\sqrt{1 + \cos(z)}$
d. $\sqrt{1 - \sin(z)}$ e. $-\frac{1}{2}\sin(z)(1 + \cos(z))^{-1/2}$

[b.] $\sqrt{1 + \cos(z)}$

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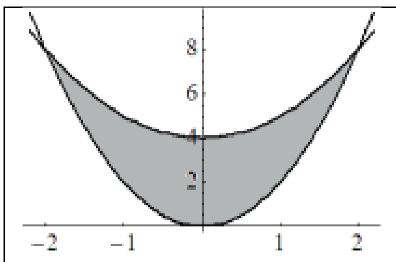
3. Which of the following integrals gives the area of the region bounded by the curves $y = 2x^2$ and $= x^2 + 4$?
a. $\int_0^2 (3x^2 + 4)dx$ b. $\int_0^2 (x^2 - 4)dx$ c. $\int_{-2}^2 (3x^2 + 4)dx$ d. $\int_{-2}^2 (x^2 - 4)dx$ e. $\int_{-2}^2 (4 - x^2)dx$



Solving simultaneously, we see that the curves intersect when $2x^2 = x^2 + 4$, so $x^2 = 4$, and $x = \pm 2$. The first curve is on the bottom between these values as sketched, so the area is given by $\int_{-2}^{2}\left(x^2 + 4 - (2x^2)\right)dx = \int_{-2}^{2}(4 - x^2)dx$ [e.]

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4. A spring's constant is $k = 6$ ft/lb. How much work (in ft-lbs) does it take to stretch it to 3 ft beyond its natural length?
a. 1/3 b. 1 c. 3 d. 9 e. 27

Force is given by Hooke's law, $f(x) = kx$, so here, $f(x) = 6x$. We integrate with respect to distance stretched to find work, so $W = \int_0^3 6x\, dx = 3x^2\big|_0^3 = 27$. [e.]

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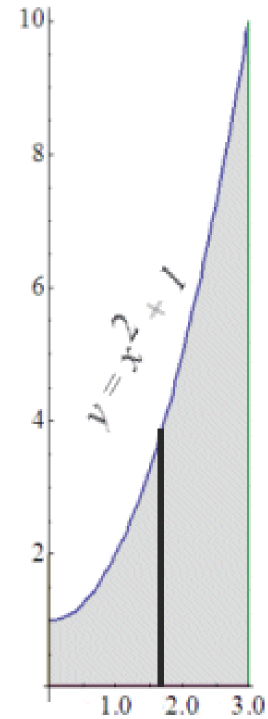
5. Which expression gives the volume of the solid formed by rotating the pictured shaded region (bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 3$) about the x-axis?



a. $\int_0^3 \pi(x^2+1)^2 dx$
b. $\int_0^1 \pi(y-1)dy$
c. $\int_0^3 \pi(x^4+1)dx$
d. $\int_0^{10} \pi(y-1)^2 dy$
e. $\int_0^1 \pi 3^2 dy + \int_1^{10} \pi\left(3^2-(y-1)\right)dx$

Using disks from rotating thin rectangles as drawn, the volume is given by $\int_0^3 \pi(x^2+1)^2 dx$ [a.]

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6. The best u for evaluating the integral $\int_0^1 \frac{x}{e^{2x}}dx$ using integration by parts (writing the integral as $\int u\, dv$) is

a. $2x$ b. x c. e^{2x} d. e^{-2x} e. $\ln(2x)$

Letting $u = x$ and $dv = e^{-2x}dx$ leads to $du = dx$ and $v = -\frac{1}{2}e^{-2x}$ so that $\int u\, dv = uv - \int v\, du = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}dx$. [b.]

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7. A 20 foot sailboat's deck is measured across every 5 ft from bow to stern resulting in the following 5 measurements: 0 ft, 6 ft, 10 ft, 12 ft, 8 ft. Using Simpson's rule with $n = 4$ to approximate the area of the deck gives, to the nearest square ft:

a. 140 b. 153 c. 160 d. 167 e. 180

We have $S_4 = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = \frac{5}{3}(0 + 4(6) + 2(10) + 4(12) + 8) = \frac{5}{3}(100) \cong 167$ [d.]

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8. For what value of c will the function f below be a probability density function?

$$f(x) = \begin{cases} \frac{c}{x^3}, & \text{if } x > 1 \\ 0, & \text{if } x \le 1 \end{cases}$$

a. ¼ b. ½ c. 1 d. 2 e. 4

To be a probability density function, we need $1 = \int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} cx^{-3}dx = -\frac{c}{2x^2}\Big|_1^{\infty} = \frac{c}{2}$. So $c = 2$. [d.]

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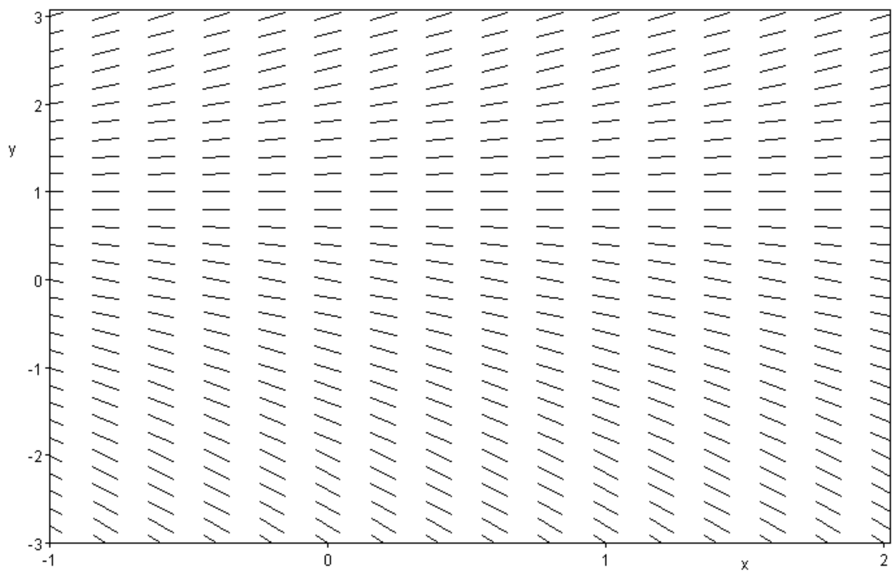
9. Given the attached direction field for a differential equation (note that the window view of the $xy-$ plane goes from -1 to 2 in the x direction, and from -3 to 3 in the y direction), if the initial condition is $y(0) = 1$ then which value below is closest to the value of the solution at $x = 1$?



a. -2 b. -1 c. 0 d. 1 e. 2

For a solution starting out at $y(0) = -1$, the y values decrease and $y(1) < -2$. **[a.]**

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10. Recall that the voltage drops across resistors, capacitors, and inductors are given by $E_R = IR$, $E_C = Q/C$, and $E_L = LI'$ respectively, and that the derivative of charge is current (i.e., $Q' = I$). What then is the differential equation describing current for a circuit consisting of a 4 henry inductor, an 8 ohm resistor, and a constant voltage of 16 volts?
a. $8I' + 4I = 16$ b. $4I' + 8I = 16$ c. $I'/4 + 8I = 16$ d. $8I' + I/4 = 16$ e. $I'/4 + I/8 = 16$

The voltage drop across the 4 henry inductor is $4I'$ and across the 8 ohm resistor is $8I$. Since the total voltage drop should equal the EMF of 16 volts, we have $4I' + 8I = 16$. **[b.]**

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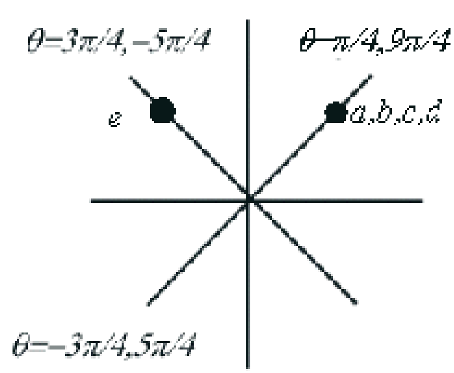
11. Five sets of polar coordinates (r, θ) are given. Four of them all represent the same point. Which set of polar coordinates represents a DIFFERENT point from the other four?

a. $\left(2, \frac{\pi}{4}\right)$ b. $\left(-2, -\frac{3\pi}{4}\right)$ c. $\left(2, \frac{9\pi}{4}\right)$ d. $\left(-2, \frac{5\pi}{4}\right)$ e. $\left(2, -\frac{5\pi}{4}\right)$	Per the figure, e is different. **[e.]** 

12. The sequence with nth term $a_n = \frac{4n^2-3}{2n^2+1}$

a. converges to -3 b. converges to $1/2$ c. converges to 2 d. converges to 4 e. diverges	Dividing to and bottom by n^2 gives $a_n = \frac{4-\frac{3}{n^2}}{2+\frac{1}{n^2}}$ which converges to 2. So c.

13. Find the sum of the geometric series $\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}} = \frac{1}{4} + \frac{3}{4^2} + \frac{3^2}{4^3} + \cdots$

a. $\frac{1}{5}$ b. $\frac{3}{4}$ c. 1 d. $\frac{4}{3}$ e. 4	The first term is $a = 1/4$ and the ratio is $r = 3/4$ so the sum is $\frac{a}{1-r} = \frac{1/4}{1/4} = 1$. c.

14. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$ is

a. 0 b. 1/3 c. 1 d. 3 e. ∞

Using the ratio test, we get convergence when the limit of $|a_{n+1}/a_n| = \frac{\frac{|x-1|^{n+1}}{3^{n+1}}}{\frac{|x-1|^n}{3^n}} = \frac{|x-1|}{3}$ as $n \to \infty$ is less than 1. And so that's for $|x-1| < 3$. So $R = 3$. d.

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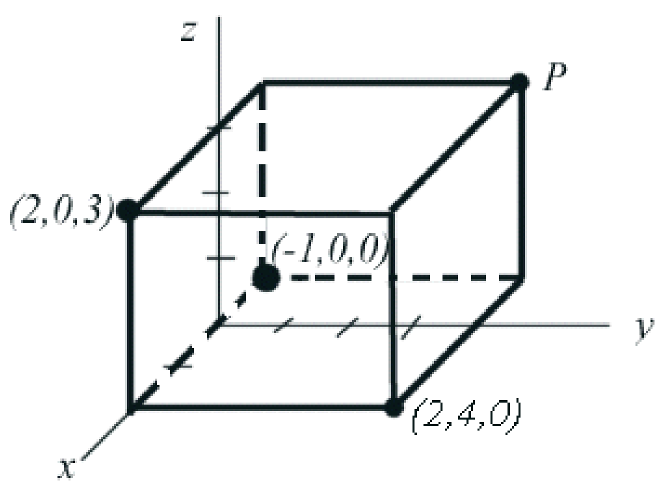
15. The Maclaurin series for e^{4x} starts as

a. $1 + 4x + 8x^2 + \frac{32}{3}x^3 + \cdots$ b. $e^{4x} + 4xe^{4x} + 8x^2e^{4x} + \cdots$ c. $1 + 4x + 16x^2 + 64x^3 + \cdots$ d. $1 + 4x + 8x^2 + \frac{64}{3}x^3 + \cdots$ e. $1 - 4x + 8x^2 - \frac{64}{3}x^3 + \cdots$	Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, substituting $4x$ for x gives $e^{4x} = 1 + 4x + \frac{16x^2}{2} + \frac{64x^3}{6} + \cdots = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \cdots$, so a.

16. For the rectangular box as drawn and labeled, the coordinates of the corner P are

a. $(-1,4,3)$
b. $(-1,3,4)$
c. $(3,4,-1)$
d. $(4,3,-1)$
e. $(3,-1,4)$

For $P(x,y,z)$, $(-1,0,0)$ has the same x value, so $x = -1$, $(2,4,0)$ has the same y value, so $y = 4$, and $(2,0,3)$ has the same z value, so $z = 3$. Hence $P = (-1,4,3)$, a.



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17. Given $\boldsymbol{a} = -\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} = \langle -1,4,1 \rangle$ and $\boldsymbol{b} = -2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k} = \langle -2,1,-3 \rangle$ find the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, $\boldsymbol{proj_a b}$.

a. $\langle -\frac{6}{14}, \frac{3}{14}, -\frac{9}{14}\rangle$ b. $\langle \frac{6}{14}, \frac{3}{14}, \frac{9}{14}\rangle$ c. $\langle \frac{6}{14}, -\frac{3}{14}, \frac{9}{14}\rangle$ d. $\langle \frac{1}{6}, \frac{4}{6}, \frac{1}{6}\rangle$ e. $\langle -\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\rangle$	$\boldsymbol{proj_a b} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\lvert\boldsymbol{a}\rvert}\right)\frac{\boldsymbol{a}}{\lvert\boldsymbol{a}\rvert} = \frac{2+4-3}{1+16+1}\langle -1,4,1\rangle = \frac{1}{6}\langle -1,4,1\rangle = \langle -\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\rangle$ e.

18. For $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ the standard unit basis vectors, which of the following is a vector?

a. $\boldsymbol{i}\cdot\boldsymbol{j}$ b. $(\boldsymbol{i}\times\boldsymbol{j})\cdot\boldsymbol{k}$ c. $\lvert\boldsymbol{i}+2\boldsymbol{k}\rvert$ d. $\boldsymbol{j}\times\boldsymbol{k}$ e. $\boldsymbol{i}\cdot\boldsymbol{i}+\boldsymbol{j}\cdot\boldsymbol{k}$ d.

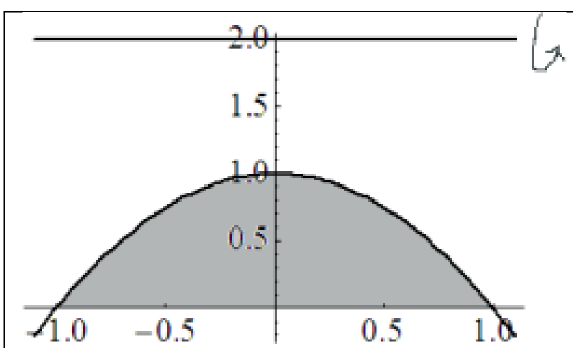
19. Which of the following is an equation of the plane determined by points $P(2,4,7)$, $Q(1,3,8)$, and $R(-3,3,7)$?

a. $x-5y-4z=-46$ b. $2x+4y+7z=0$ c. $2x+y-3z=7$ d. $2x+3y+z=1$ e. $2x+y-z=1$	Method 1: Test the points in the equations. Point P satisfies a. and e. only. Q doesn't satisfy e., so a.	Method 2: $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle -1,-1,1\rangle\times\langle -5,-1,0\rangle=\langle 1,-5,-4\rangle$. So the plane has equation $(x-2)-5(y-4)-4(z-7)=0$ which simplifies to a.

20. Parametric equations for the line through points $(2,3,1)$ and $(4,2,2)$ are

a. $x=2+2t,\quad y=3-t,\quad z=1+t$ b. $x=-2+2t,\quad y=-3-t,\quad z=-1+t$ c. $x=2+2t,\quad y=1-3t,\quad z=1+t$ d. $x+3y+z=12$ e. $(x-2)+(y-3)+(z-1)=2$	Method 1: Note that d and e are equations for planes not lines. Solving for $x=2$ we see that only line a. goes through point 1. a.	Method 2: The displacement vector determined by the points is $\langle 2,-1,1\rangle$. The standard parametric equations are then a.

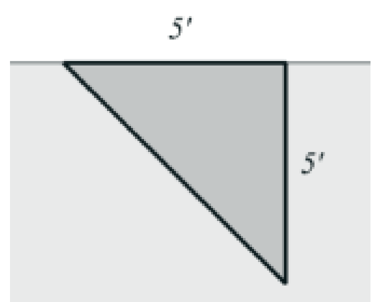
21. Let R be the region bounded by the $x-$ axis and the parabola $y=1-x^2$ as drawn. Find the volume of the solid generated when R is revolved about the horizontal line $y=2$.

	Using washers, the volume is given by $V=\int_{-1}^{1}\pi 2^2-\pi\left(2-(1-x^2)\right)^2\,dx=$ $\pi\int_{-1}^{1}3-2x^2-x^4\,dx=\frac{64\pi}{15}\cong 13.404$

22. A chain that weights 3 lbs/ft is used to lift a 200 lb anchor 60 ft to the deck. Find the total work done.
For the anchor only, the work is $200\cdot 60=12{,}000$ft-lbs. For the chain the work is $\int_0^{60}3x\,dx=5{,}400$ft-lbs. So the total work is $17{,}400$ft-lbs.

23. Find the hydrostatic force on one face of a plate as drawn – it's a vertical isosceles right triangle with shortest sides of length 5 ft and one of those short sides at the surface of the water. Assume that the weight of water is 62.5 lbs/ft^3.

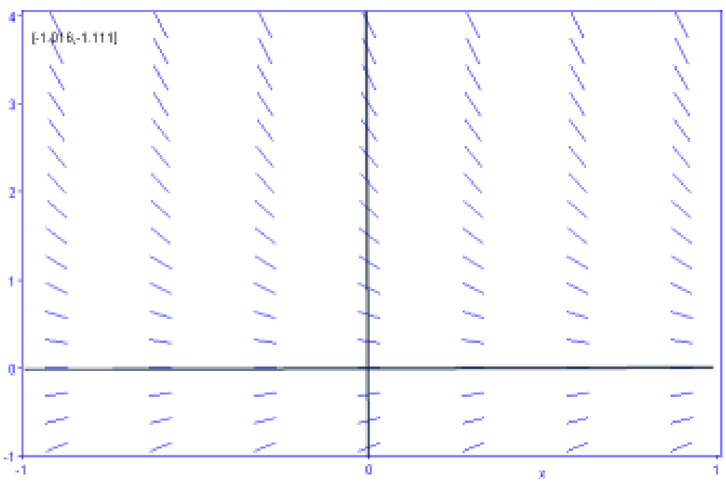
$62.5\int_0^5 x(5-x)dx = 15625/12 =$ $\boxed{1302.08\overline{3}\text{ lbs}}$



24. Consider the differential equation $y' = -2y$ with initial condition $y(0) = 3$.
a. Sketch the direction field, including at least one slope line in each quadrant.
b. Use Euler's method with two steps of size $1/4$ to approximate $y(1/2)$.
c. Use the separable equation method to solve the equation and find $y(1)$ exactly, showing your work.
a.

n	x_n	y_n	$\Delta y = h\cdot y' = -y_n/2$
0	0	3	$-3/2$
1	0.25	3/2	$-3/4$
2	0.5	3/4	

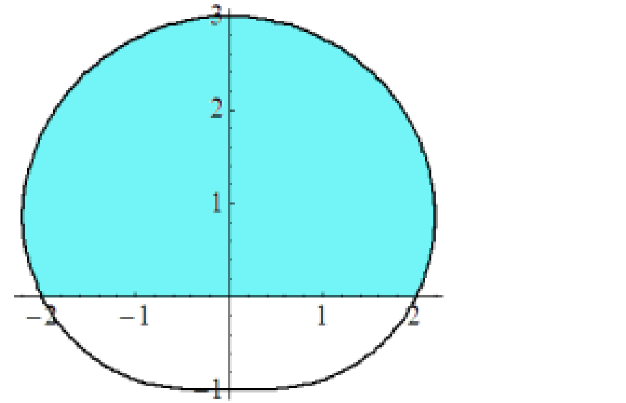
So $y(0.5) \cong \boxed{0.75}$



b.

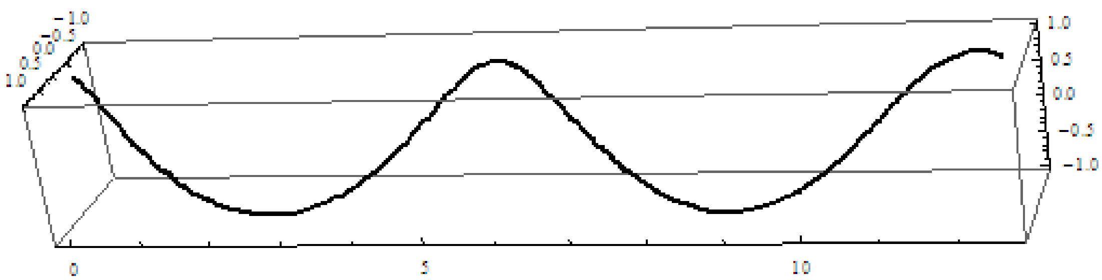
c. $\frac{dy}{dx} = -2y$, so $\int \frac{dy}{y} = \int -2\,dx$, and $\ln(y) = -2x + C$, so $y = e^{-2x+C} = A\,e^{-2x}$. For $y(0) = 3$, we get $A = 3$ and $y = 3e^{-2x}$. And $y(1) = \boxed{3/e^2}$

25. Sketch the curve with polar equation $r = 2 + \sin(\theta)$ and shade the region inside the curve but above the $x-$ axis. Find the area of the shaded region. (Hint: use your calculator to integrate.)



$A = \frac{1}{2}\int_0^{\pi}(2+\sin(\theta))^2\,d\theta = \boxed{4 + 9\pi/4 \cong 11.069}$

26. Consider the vector equation $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$.
a. Sketch the curve it describes.
b. Find parametric equations for the tangent line to the curve at the point $(1, \pi/2, 0)$.



b. $\boldsymbol{r}'(t) = \langle \cos(t), 1, -\sin(t) \rangle$, and the given point is $\boldsymbol{r}(\pi/2)$. Since $\boldsymbol{r}'(\pi/2) = \langle 0,1,-1 \rangle$, the line is given by $\boxed{x = 1,\ y = \pi/2 + t,\ \ z = -t}$

27. Evaluate:

a. $\int x\cos(3x)\,dx$ Letting $u = x, dv = \cos(3x),\ du = dx, v = \frac{1}{3}\sin(3x)$. So $\int x\cos(3x)\,dx = \int u\,dv = uv - \int v\,du = \frac{1}{3}x\sin(3x) - \int \frac{1}{3}\sin(3x)\,dx = \boxed{\frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C}$

b. $\int \frac{x+25}{x^2-5x+4}dx$ Clearing denominators of $\frac{x+25}{x^2-4x+5} = \frac{A}{x-4} + \frac{B}{x-1}$ gives $x + 25 = A(x-1) + B(x-4)$ and taking $x = 1$ and $x = 4$ gives $B = -\frac{26}{3}$ and $A = \frac{29}{3}$. So $\int \frac{x+25}{x^2-5x+4}dx = \int \frac{29}{3}\frac{1}{x-4} - \frac{26}{3}\frac{1}{x-1}dx =$ $\boxed{\frac{29}{3}\ln|x-4| - \frac{26}{3}\ln|x-1| + C}$

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28. Given $\boldsymbol{a} = -\boldsymbol{j} - 2\boldsymbol{k} = \langle 0,-1,-2 \rangle$ and $\boldsymbol{b} = 3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k} = \langle 3,1,-4 \rangle$ find
a) $|\boldsymbol{a}| = \sqrt{5}$
b) $3\boldsymbol{b} - 2\boldsymbol{a} = \langle 9,5,-8 \rangle$
c) $\boldsymbol{a}\cdot\boldsymbol{b} = 7$
d) $\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -1 & -2 \\ 3 & 1 & -4 \end{vmatrix} = \langle 6,-6,3 \rangle$ (check perp to **a** and **b**)

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29. Find the Taylor series for $f(x) = \ln(x)$ centered at a =2. Write out the first 4 non-zero terms.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$c_n = f^{(n)}(2)/n!$	$c_n(x-a)^n$
0	$\ln(x)$	$\ln(2)$	$\ln(2)$	$\ln(2)$
1	$1/x$	½	$1/2$	$\frac{1}{2}(x-2)$
2	$-1/x^2$	$-1/4$	$-1/(4\cdot 2!)$	$-\frac{1}{8}(x-2)^2$
3	$2/x^3$	$2/8$	$2/(8\cdot 3!)$	$\frac{1}{24}(x-2)^3$
4	$-3!/x^4$	$-3!/16$	$-3!/(16\cdot 4!)$	$-\frac{1}{64}(x-2)^4$
n	$(-1)^{n+1}(n-1)!/x^n$	$(-1)^{n+1}(n-1)!/2^n$	$(-1)^{n+1}/(n2^n)$	$\frac{(-1)^{n+1}}{n2^n}(x-2)^n$

So we get $\ln(2)+\frac{1}{2}(x-2)-\frac{1}{8}(x-2)^2+\frac{1}{24}(x-2)^2-\cdots$.

30. Prove the Mean Value Theorem for Integrals (If f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$) by applying the Mean Value Theorem for derivatives (If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a number c in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$), to the function $F(x)=\int_a^x f(t)\,dt$.

Proof. We know that for f is continuous on $[a,b]$ the function $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) with derivative $f(x)$. So by the Mean Value Theorem for derivatives, there is a number c in (a,b) such that $f(c)=F'(c)=\frac{F(b)-F(a)}{b-a}$. Since $F(a)=0$, this can be rewritten as

$f(c)=\frac{F(b)}{b-a}=\frac{1}{b-a}\int_a^b f(t)\,dt$ and we've shown the Mean Value Theorem for integrals.